$$f''(0) = (1/\bar{\delta})\ (2 + \lambda/6) \quad (26)$$

$$\Delta = \bar{\delta}(\tfrac{3}{10} - \lambda/20) \quad (27)$$

where

$$M = \sigma(d\, U_1/d\, \sigma)/U_1 \quad (28)$$

For $M = 0$, $\bar{\delta} = 3.17$, $f''(0) = 0.7846$, and $\Delta = 0.8739$. Ting gives $f''(0) = 0.7866$ and $\Delta = 0.8695$. For $M = \frac{1}{3}$ one obtains $f''(0) = 0.7411$ and $\Delta = 0.9715$. Comparison with Ting's solution for $M = 0$ shows surprisingly good accuracy for the approximate method. The variation of $f''(0)$ for $0 < M < \frac{1}{3}$ is quite small. The corresponding variation of Δ, and hence the profile (f', f'') is somewhat greater.

Thus Ting's solution is only one member of a similar family of flows. In fact, Ting's solution is also valid for bodies that attain a finite width as $s \to \infty$.

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Comments on "Stability of Damped Mechanical Systems" and a Further Extension

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PRINGLE'S note[1] yields an immediate proof of what I have elsewhere[2] called the Kelvin-Tait-Chetaev theorem. Also it generalizes the theorem in a different way from the generalization given in Ref. 2. Furthermore, for linear systems, Pringle's note suggests still another refinement. Since these results are very useful for satellite attitude dynamics, it may be helpful to compare and illustrate them by means of a simple example and to give a proof of the further refinement.

Consider the following system:

$$\begin{aligned} m_1\ddot{x}_1 + d_0\dot{x}_1 + d_1(\dot{x}_1 - \dot{x}_2) + g\dot{x}_2 + k_1x_1 &= 0 \\ m_2\ddot{x}_2 - d_1(\dot{x}_1 - \dot{x}_2) + d_0\dot{x}_2 - g\dot{x}_1 + k_2x_2 &= 0 \end{aligned} \quad (1)$$

The terms containing d_0 and d_1 are damping terms; those containing g arise from gyroscopic effects. System (1) is characteristic of larger systems that arise in small-angle attitude-control studies of the form:

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0 \quad (2)$$

where $\mathbf{A}$ is a symmetric, positive definite inertia matrix, $\mathbf{D}$ is a symmetric damping matrix, $\mathbf{G}$ is a skew-symmetric gyroscopic matrix, and $\mathbf{K}$ is a symmetric stiffness matrix. With little loss in generality, we shall assume throughout that $\mathbf{K}$ has no zero eigenvalues, so that in (1) $k_1k_2 \neq 0$.

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To investigate the stability of (2), one could calculate the roots s of the system's characteristic polynomial:

$$|s^2\mathbf{A} + s(\mathbf{D} + \mathbf{G}) + \mathbf{K}| = 0 \quad (3)$$

But if $\mathbf{D}$ is positive definite, the *KTC* (Kelvin-Tait-Chetaev) theorem gives the following simplification:

Theorem A: When $\mathbf{D}$ *is positive definite, Eq. (3) has all its roots in the open left half-plane, if* $\mathbf{K}$ *has all positive eigenvalues and at least one root in the open right half-plane, and if* $\mathbf{K}$ *has any negative eigenvalues.*

Thus, when $\mathbf{D}$ is positive definite, the character of the roots of (3) is determined completely by the character of the roots of $|\mathbf{K} - s\mathbf{I}| = 0$, where $\mathbf{I}$ is the unit matrix. Furthermore, in many cases the coordinates are coupled only via damping and gyroscopic terms, so that not only is $|\mathbf{K} - s\mathbf{I}| = 0$ of degree n instead of $2n$, but it is of much simpler structure than Eq. (3). For instance, in the example (1), $\mathbf{D}$ is clearly positive definite for $d_0 > 0$, $d_1 > 0$, and $\mathbf{K}$ has only the diagonal elements k_1, k_2. If k_1, k_2 are both positive, the KTC theorem says that all four roots lie in the open left half-plane; if k_1 or k_2 are negative, one or more roots is in the open right half-plane.

In Ref. 2, the KTC theorem is sharpened as follows:

Theorem B: When $\mathbf{D}$ *is positive definite, Eq. (3) has no roots on the imaginary axis and as many roots in the right half-plane as there are negative eigenvalues of* $\mathbf{K}$. Thus in the simple example if $d_0 > 0$, $d_1 > 0$, and if $k_1 < 0$, $k_2 < 0$, we can conclude that there are two roots in the right half-plane, two roots in the left half-plane, and no roots on the imaginary axis.

However, in attitude-control systems, Theorems A and B are often inapplicable. There are typically no "ground" dampers between the satellite parts and the reference frame; energy is dissipated only by relative motion between satellite parts. In this case, the damping matrix $\mathbf{D}$ in (2) is only positive semidefinite rather than positive definite, that is, the power dissipation function $P = -(\dot{\mathbf{x}}, \mathbf{D}\dot{\mathbf{x}})$ may be zero as well as negative for some choices of $\dot{\mathbf{x}}$. For example, if the "ground" dampers in (1) are absent, $d_0 = 0$, then $P = -d_1(\dot{x}_1 - \dot{x}_2)^2$, which is zero for $\dot{x}_1 = \dot{x}_2$.

As Pringle has pointed out in a private communication, the assumption of a positive definite $\mathbf{D}$ is too restrictive. In the design of attitude-control systems, one strives not for a positive definite damping matrix but rather for damping that affects the entire system, so that any motion induces energy dissipation. It is perhaps helpful to coin the term *pervasive* for this kind of damped system. More specifically, form the scalar product of Eq. (2) with $\dot{\mathbf{x}}$ to get

$$(d/dt)[\tfrac{1}{2}(\dot{\mathbf{x}}, \mathbf{A}\dot{\mathbf{x}} + \mathbf{x}, \mathbf{K}\mathbf{x})] = P = -(\dot{\mathbf{x}}, \mathbf{D}\dot{\mathbf{x}}) \quad (4)$$

Then we define a pervasive power dissipation P as follows: P is pervasive if it is nonpositive and can be zero for all $t > 0$ only when the system is at the equilibrium point for $t > 0$. In (1) damping is pervasive for $d_0 = 0$ and $g \neq 0$. To show this, we note that when $P = -d_1(\dot{x}_1 - \dot{x}_2)^2 \equiv 0$, we have $x_1 = x_2 + C$, where C is a constant. But Eqs. (1) then become

$$m_1\ddot{x}_1 + g\dot{x}_1 + k_1x_1 = 0 \qquad m_2\ddot{x}_1 - g\dot{x}_1 + k_2x_1 = C$$

which can be satisfied only when $x_1 = C = x_2 = 0$. On the other hand, if $g = 0$ and $k_1/m_1 = k_2/m_2$, the damping is not pervasive; m_1 and m_2 can remain at a fixed distance apart in an oscillation with zero power dissipation. Note that pervasiveness is a property of both the structure of P and the structure of the differential equations. In general, an argument such as the preceding is needed to show pervasiveness.

The generalization of the idea of pervasive damping is used by Pringle. The bracketed term in Eq. (4) is the Hamiltonian of system (2). In a general dynamical system, instead of (4) we have an equation of the form $dH/dt = P$, where both H, the Hamiltonian, and P are functions of the generalized coordinates and momenta q_i, p_i. We assume that the origin in the (q_i, p_i) space is an equilibrium point and adopt the previous definition of pervasiveness in the general case. Then corollary 3 of Pringle's note is:

Theorem C: If the Hamiltonian H is independent of time and if the power dissipation function P is pervasive, then the dynamical system is asmyptotically stable if H is positive definite in a region S around the origin in the (q_i, p_i) space and unstable if H is indefinite or negative definite in S.

Theorem C gives an almost immediate proof of the KTC theorem, Theorem A, which is simpler than the proofs in Refs. 1 and 2. If $\mathbf{D}$ is positive definite, $P \equiv 0$ implies $\dot{\mathbf{x}} \equiv 0$. But then by (2), $\mathbf{Kx} = 0$, and hence $\mathbf{x} = 0$, since we have assumed that $\mathbf{K}$ has no zero eigenvalues. Therefore P is pervasive. Since $\mathbf{A}$ is assumed positive definite, the Hamiltonian $H = \frac{1}{2}[\dot{\mathbf{x}}, \mathbf{A}\dot{\mathbf{x}} + \mathbf{x}, \mathbf{Kx}]$ is positive definite with respect to $q_i = x_i$, $p_i = \dot{x}_i$ if $\mathbf{K}$ has all positive eigenvalues and indefinite if $\mathbf{K}$ has some negative eigenvalues, which proves Theorem A.

Moreover, Theorem C is not restricted to linear systems. For example, for $d_0 = 0$, $g \neq 0$, damping terms in (1) of the form $d_1(\dot{x}_1 - \dot{x}_2)$ or $d_1|\dot{x}_1 - \dot{x}_2|(\dot{x}_1 - \dot{x}_2)$ both give a pervasive power function P. In either case, the stability of system (1) would be completely determined by the signs of the k's.

Likewise, Theorem C suggests that Theorem B will remain valid if the assumption of pervasiveness is used. This is indeed the case; we have:

Theorem D: If in system (2) $P = -(\dot{\mathbf{x}}, \mathbf{D}\dot{\mathbf{x}})$ is pervasive, Eq. (3) has no roots on the imaginary axis and as many roots in the closed right half-plane as there are negative eigenvalues of $\mathbf{K}$.

The proof is very similar to the proof of Theorem II of Ref. 2 and will only be sketched here. Let $\mathbf{x} = \mathbf{V} \exp st$, where $\mathbf{V}$ is an eigenvector satisfying the matrix equation $(s^2\mathbf{A} + s\mathbf{D} + s\mathbf{G} + \mathbf{K})\mathbf{V} = 0$. First, one shows that in a pervasive system, $(\mathbf{V}^*, \mathbf{DV}) = 0$ implies that $\mathbf{V} = 0$ [$\mathbf{V}^*$ is the complex conjugate of $\mathbf{V}$ and $(\mathbf{V}^*, \mathbf{DV}) = \Sigma V_i^* V_j D_{ij}$]. To do this, let $\mathbf{V} = \mathbf{Q}\boldsymbol{\xi}$, where $\mathbf{Q}$ is a matrix[4] diagonalizing $\mathbf{D}$ into the form: $(d_{11}, d_{22}, \ldots, d_{mm}, 0, 0, \ldots)$, and let $\mathbf{x} = \mathbf{V} \exp st + \mathbf{V}^* \exp s^*t$. Then we have the following chain of arguments, where $\rightarrow$ denotes "implies":

$(\mathbf{V}^*, \mathbf{DV}) = 0 \rightarrow \xi_1 = \xi_2 = \ldots = \xi_m = 0 \rightarrow (\mathbf{V}, \mathbf{DV}) = (\mathbf{V}^*, \mathbf{DV}^*) = 0 \rightarrow \mathbf{x}, \mathbf{Dx} = 0 \rightarrow \mathbf{x} = 0 \rightarrow \mathbf{V} = 0$. Hence, for any nonzero eigenvector, $(\mathbf{V}^*, \mathbf{DV}) > 0$.

Next we define $\hat{\mathbf{D}}$ and $\hat{\mathbf{G}}$ by $\hat{\mathbf{G}} = r\mathbf{G}$ and $\hat{\mathbf{D}} = r\mathbf{D} + (1 - r)\mathfrak{D}$, where $0 \leq r \leq 1$, and $\mathfrak{D}$ is a diagonal matrix consisting of ones where the corresponding diagonal elements of $\mathbf{D}$ are zero and of the diagonal elements themselves where they are nonzero. As in Ref. 2, when r goes from 0 to unity, the roots of the characteristic equation do not cross the imaginary axis, and hence there are as many roots in the right half-plane as there are negative eigenvalues of $\mathbf{K}$.

Finally, we note that in system (2) we may have a positive semidefinite $\mathbf{D}$ but a nonpervasive power function. For such systems Theorems A–D do not apply. However, the following result may apply (see Theorem III of Ref. 2):

Theorem E: If $\mathbf{D}$ *is positive semidefinite in system (2) Eq. (3) will have all its roots in the closed left half-plane if all the eigenvalues of* $\mathbf{K}$ *are positive, and at least one root in the open right half-plane if all the eigenvalues of* $\mathbf{K}$ *are negative.*

Thus by Theorem E if $d_0 = g = 0$ in (1) and $k_1/m_1 = k_2/m_2$, all of the four roots are on the imaginary axis or in the left half-plane if $k_1 > 0$, and there is at least one root in the open right half-plane if $k_1 < 0$.

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Chemical Nonequilibrium Studies in Supersonic Nozzle Flows

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Introduction

In many cases an accurate analysis of nozzle flow fields requires the inclusion of two-dimensional effects and finite rate chemistry. The theoretical model for incorporating chemical reactions in two-dimensional inviscid supersonic nozzle flows is available[1] and numerical solutions have been obtained.[2–4] In Ref. 2 an approximation is presented wherein the nonequilibrium terms are neglected along Mach lines but retained along streamlines. The justification for this assumption was that the Mach line characteristics are used only to determine the pressure and geometry, and the pressure does not vary significantly between the two extremes of equilibrium and frozen flow. The approximation, however, leads to a small reduction in pressure, which, in many cases, is of the same order of magnitude as the reduction in pressure due to nonequilibrium effects. Hence there exists the possibility of obscuring the nonequilibrium effects in one of the most important flow field parameters for thrust and drag calculations, namely the wall pressure distribution. The numerical procedure adopted in Ref. 4 is unrelated to the characteristics, and hence the authors indicate caution in dealing with flows where wave phenomena are an important feature. Waves are always an important feature in nozzle flows. For example, even simple conical nozzles can lead to shock formation,[5,6] and many propulsion and wind-tunnel nozzles are characterized by a rapid initial expansion section.[7] Some form of precise differencing technique is required in all of the preceding methods when the flow parameters are near their equilibrium values. Thus complicated and lengthy numerical integration techniques are required to achieve an acceptable degree of accuracy. A review of some of the past experimental and analytical studies with one-dimensional flow fields[8–11] suggests that a two-dimensional analysis should be

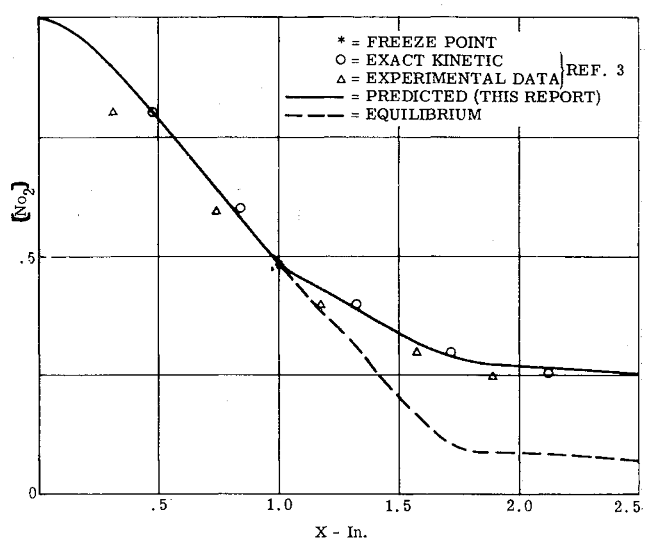

Fig. 1 Comparison with experimental data.

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